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Finding the normal force in planar non-uniform circular motion using polar coordinates

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This could be seen as an undergraduate-physics-level post.

In this post, we will derive an expression for the normal force on a uniform mass which is in planar non-uniform circular motion using polar coordinates. Finding this expression is enormously useful to calculate under which circumstances a mass would be slung off its orbital path. Of course, there are numerous situations for which we should be able find the normal force. Here, we will look at a system as shown in Figure 1. Sometimes, obtaining an expression in terms of the variables given is not straightforward. You will find a useful trick in step 7 to arrive at an expression in terms of a simple θ instead of its secondary-order derivative $\ddot{\theta}$ which we initially obtain.

Notation

We will apply Newton's notation (the dot notation) whenever possible as this is the most compact form. For instance, if $\mathbf{x}$ is a vector, then its first-order and its second-order derivative with respect to time t are denoted by

$$\dot{\mathbf{x}} \text{ and } \ddot{\mathbf{x}},$$

respectively. Where needed, in order to state explicitly that we are dealing with a time-derivative and to help in solving an time-integral for example, we will use Leibniz's notation, i.e.

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \text{ and } \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}.$$

Assignment

Look at the system as sketched in Figure 1. Imagine we stand in front of this system. Mass m is attached to a model string. At $t = 0$, it rests at level with the centre of the cylinder with radius R with the string draped over the top. A constant force $\mathbf{P}$ pulls the string downwards. At a later time t, mass m has slid over the top with a coefficient of friction μ. Let θ denote the angle between its initial and its current position, subtended at the centre of the cylinder. Calculate the normal force on m, and, hence, proof that the radius of the cylinder is irrelevant.

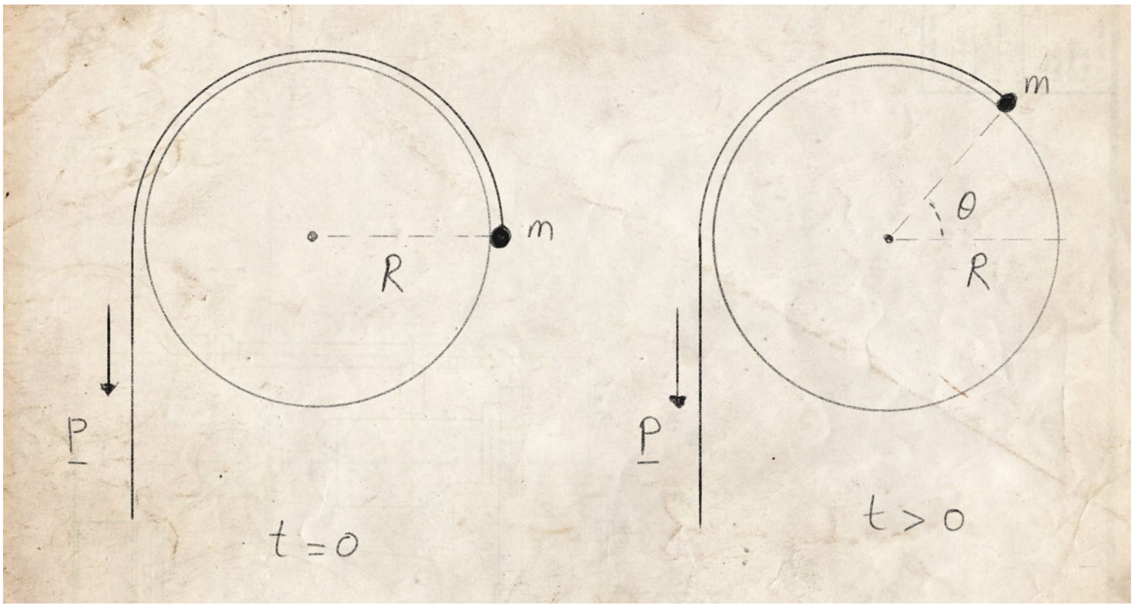

Figure 1: The system

Step 1. Force diagrams and unit vectors

It is essential to draw force diagrams and unit vectors to define the acting forces and parameters. We choose the unit vectors to be the radial and the tangential vectors. This makes calculating most forces a lot easier. This is done in Figure 2.

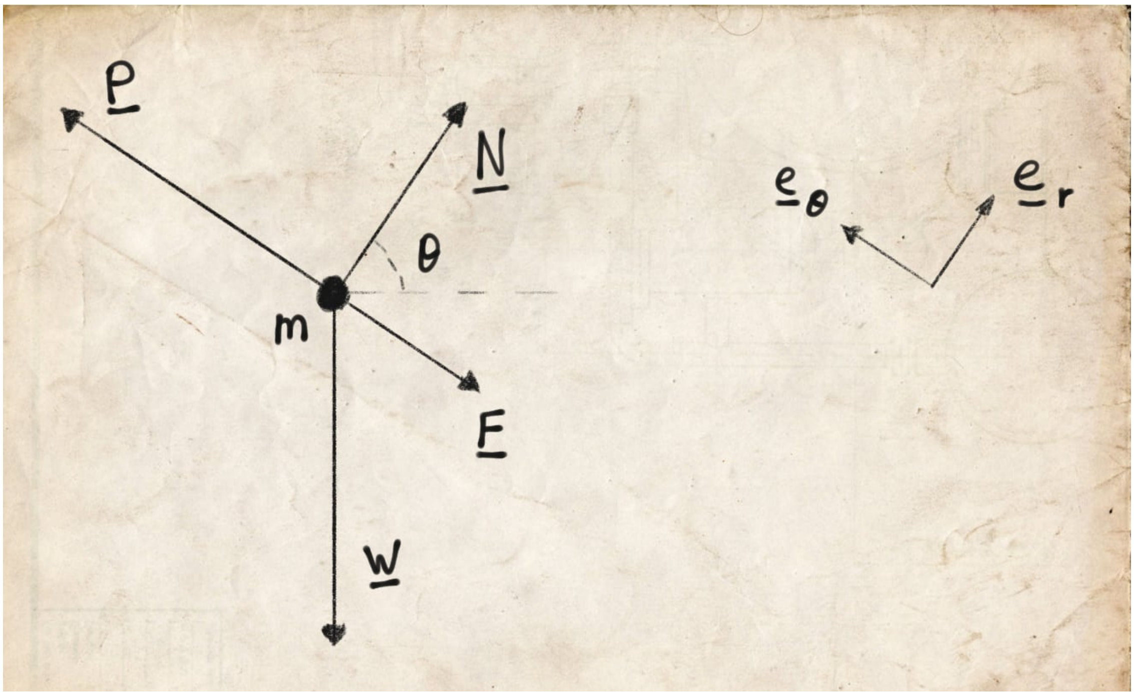

Figure 2: Force diagram and unit vectors at time $t > 0$

We identify the following forces on m:

$\mathbf{P}$ is the vector denoting the constant force pulling the model string,
$\mathbf{N}$ is the vector denoting the normal force acted on m by the cylinder,
$\mathbf{F}$ is the vector denoting the frictional force,
$\mathbf{W}$ is the vector denoting the weight of m as a result of the gravitational field of whatever planet the system is located,
$\mathbf{e}_r$ is the radial unit vector,
$\mathbf{e}_\theta$ is the tangential unit vector.

Step 2. Apply Newton's second law

As this is a dynamical system, where m is in non-uniform circular motion, we apply Newton's second law, more specifically in the following form:

$$\sum \mathbf{F} = m\ddot{\mathbf{r}}, \tag{1}$$

where $\ddot{\mathbf{r}}$ is the rate of change of the rate of change over time, that is, the second time-derivative of the displacement vector $\mathbf{r}$ of mass m. We can now easily identify the constituents of the vector sum as we did that already in Step 1. And so, equation (1) becomes

$$m\ddot{\mathbf{r}} = \mathbf{P} + \mathbf{N} + \mathbf{F} + \mathbf{W}. \tag{2}$$

Step 3. Rewrite the forces in terms of their magnitudes and unit vectors

As pulling force $\mathbf{P}$ with magnitude $|\mathbf{P}|$ acts in the direction of tangential unit vector $\mathbf{e}_\theta$, we can write for $\mathbf{P}$:

$$\mathbf{P} = |\mathbf{P}|\mathbf{e}_\theta. \tag{3}$$

Since we don't have any other information regarding this force, we leave it at that.

Normal force $\mathbf{N}$ points in the direction of radial unit vector $\mathbf{e}_r$, so, we write:

$$\mathbf{N} = |\mathbf{N}|\mathbf{e}_r. \tag{4}$$

Friction $\mathbf{F}$ is in the opposite direction of the tangential unit vector $\mathbf{e}_\theta$, so, we need to place a minus-sign in its expression. Furthermore, as (dry) friction is usually modelled by the product of the coefficient of friction and the magnitude of the normal force, we can write:

$$\mathbf{F} = \mu|\mathbf{N}|(-\mathbf{e}_\theta). \tag{5}$$

Lastly, weight is the force due to gravity, $|\mathbf{W}| = mg$, where g is the gravitational constant. However, we need to express this force in terms of its components. In this case, those components are directed parallel to the radial and tangential unit vectors. As the latter are pointed (partly) upwards, as opposed to the downwards-pointing weight, we already know that both its components carry a minus-sign, i.e. $(-\mathbf{e}_r)$ and $(-\mathbf{e}_\theta)$. What remains, is the correct expression for the magnitude of the weight in terms of its respective unit vectors.

To clearly show how we get an expression for $\mathbf{W}$ in terms of its components along the directions of $\mathbf{e}_r$ and $\mathbf{e}_\theta$, have a look at Figure 3. What you see is just the weight vector $\mathbf{W}$ from our force diagram in Figure 2, including the radial and tangential unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$. For visual clarity, we subtended them on mass m. Also added are the two component vectors in the opposite direction of the unit vectors for which we need to find expressions.

Let component vector $\mathbf{v}_r = a(-\mathbf{e}_r)$ and component vector $\mathbf{v}_\theta = b(-\mathbf{e}_\theta)$, where a and b are some magnitude value such that the vector sum of $\mathbf{v}_r$ and $\mathbf{v}_\theta$ equals $\mathbf{W}$. In other words,

$$\mathbf{W} = \mathbf{v}_r + \mathbf{v}_\theta = a(-\mathbf{e}_r) + b(-\mathbf{e}_\theta). \tag{6}$$

To find the values of the magnitude of a and b, we use the fact that the magnitude $|\mathbf{W}| = mg$. So, using high school trigonometry, we deduce that

$$a = mg\sin\theta, \tag{7}$$

$$b = mg\cos\theta. \tag{8}$$

Now, we can write $\mathbf{W}$ in terms of its components by substituting equations (7) and (8) into (6):

$$\mathbf{W} = mg\sin\theta(-\mathbf{e}_r) + mg\cos\theta(-\mathbf{e}_\theta). \tag{9}$$

And so, if we substitute equations (3), (4), (5), and (9) into equation (2), we get:

$$\begin{aligned} m\ddot{\mathbf{r}} = |\mathbf{P}|\mathbf{e}_\theta + |\mathbf{N}|\mathbf{e}_r + \mu|\mathbf{N}|(-\mathbf{e}_\theta) \\ + mg\sin\theta(-\mathbf{e}_r) + mg\cos\theta(-\mathbf{e}_\theta). \end{aligned} \tag{10}$$

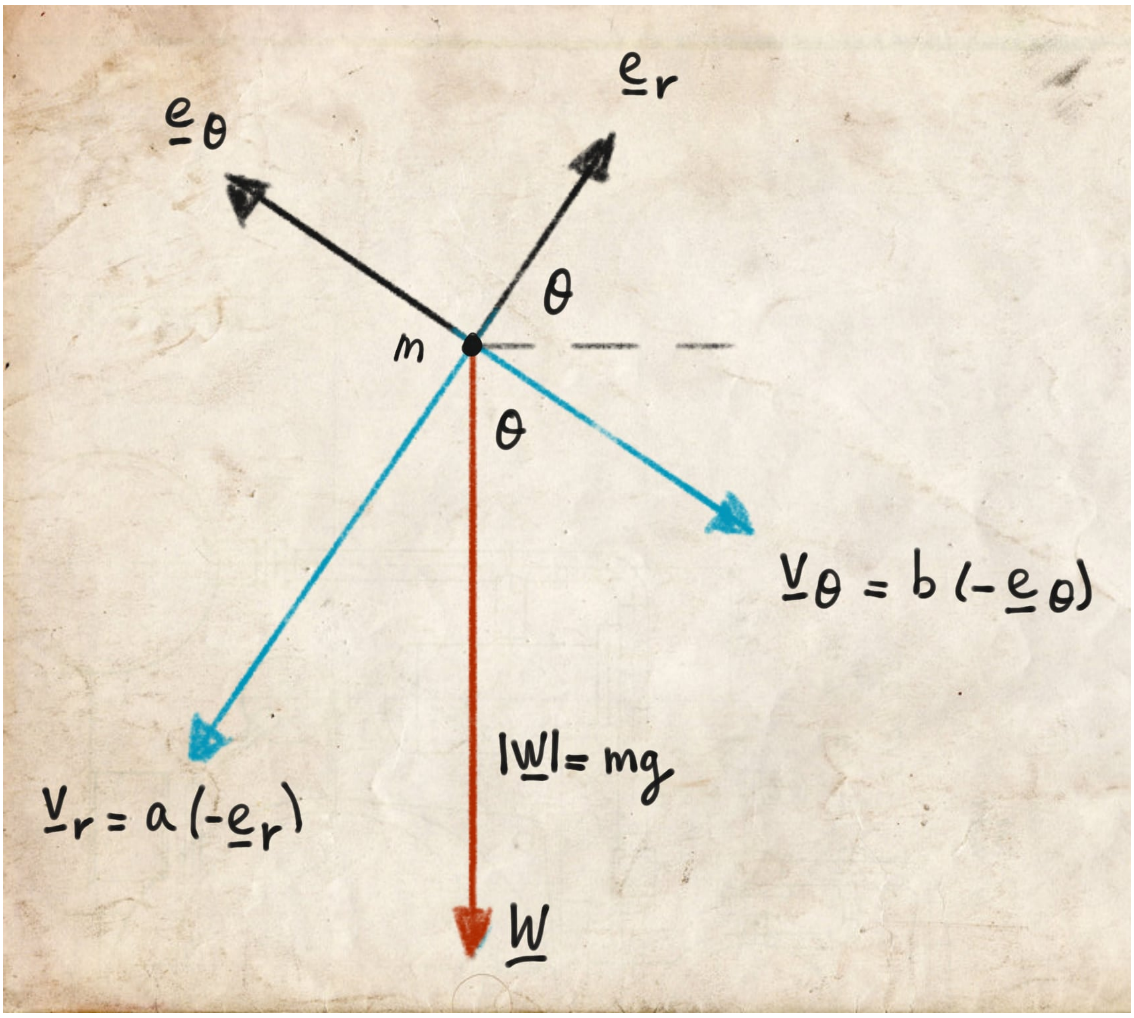

Figure 3: Finding the components of **W**

Step 4. Express the Cartesian $\ddot{\mathbf{r}}$ in polar coordinates

As we know that the expression for the second time derivative of non-uniform circular motion is

$$\ddot{\mathbf{r}} = -R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta, \tag{11}$$

where R is the radius of the circular motion, i.e. the cylinder. We proceed to substitute this into equation (10).

And so, we get

$$\begin{aligned} m(-R\dot{\theta}^2\mathbf{e}_r + R\ddot{\theta}\mathbf{e}_\theta) = |\mathbf{P}|\mathbf{e}_\theta + |\mathbf{N}|\mathbf{e}_r + \mu|\mathbf{N}|(-\mathbf{e}_\theta) \\ + mg\sin\theta(-\mathbf{e}_r) + mg\cos\theta(-\mathbf{e}_\theta), \end{aligned}$$

which, of course, after expansion, becomes

$$\begin{aligned} -mR\dot{\theta}^2\mathbf{e}_r + mR\ddot{\theta}\mathbf{e}_\theta = |\mathbf{P}|\mathbf{e}_\theta + |\mathbf{N}|\mathbf{e}_r + \mu|\mathbf{N}|(-\mathbf{e}_\theta) \\ + mg\sin\theta(-\mathbf{e}_r) + mg\cos\theta(-\mathbf{e}_\theta). \end{aligned} \tag{12}$$

Step 5. Resolve radially and tangentially

We can now resolve equation (12) into its radial and tangential components.

$$\mathbf{e}_r : -mR\dot{\theta}^2 = N - mg\sin\theta, \tag{13}$$

$$\mathbf{e}_\theta : mR\ddot{\theta} = P - \mu N - mg\cos\theta. \tag{14}$$

Step 6. Write down the equation of motion (in polar coordinates)

Rearranging equation (14), we can write down the second-order differential equation of motion:

$$\ddot{\theta} = \frac{P - \mu N - mg\cos\theta}{mR}. \tag{15}$$

While we could have solved equation (14) for N, this would still leave us with the second time-derivative of θ. Instead, we want an expression of N in terms of a simple θ. This means that we need to get rid of $\ddot{\theta}$ in some way. It is not immediately clear how equation (14) or (15) should be operated on to achieve this. However, here is a neat trick.

Step 7. The trick

Have a look at the following equation where we apply the chain rule:

$$\frac{\mathrm{d}\dot{\theta}^2}{\mathrm{d}t} = \frac{\mathrm{d}\dot{\theta}^2}{\mathrm{d}\dot{\theta}}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = 2\dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = 2\dot{\theta}\ddot{\theta}. \tag{16}$$

So, if we substitute equation (15) into (16), we get

$$\frac{\mathrm{d}\dot{\theta}^2}{\mathrm{d}t} = 2\dot{\theta}\left(\frac{P - \mu N - mg\cos\theta}{mR}\right). \tag{17}$$

If we now integrate both sides with respect to time, we get

$$\int \frac{\mathrm{d}\dot{\theta}^2}{\mathrm{d}t}\mathrm{d}t = \int 2\dot{\theta}\left(\frac{P-\mu N - mg\cos\theta}{mR}\right)\mathrm{d}t,$$

$$\dot{\theta}^2 + A = 2\int \frac{\mathrm{d}\theta}{\mathrm{d}t}\left(\frac{P-\mu N - mg\cos\theta}{mR}\right)\mathrm{d}t,$$

where A is an arbitrary constant,

$$\dot{\theta}^2 + A = 2\int \left(\frac{P-\mu N - mg\cos\theta}{mR}\right)\mathrm{d}\theta,$$

$$\dot{\theta}^2 + A = \frac{2}{mR}\int (P-\mu N - mg\cos\theta)\,\mathrm{d}\theta,$$

$$\dot{\theta}^2 + A = \frac{2}{mR}\left(P\int 1\,\mathrm{d}\theta - \mu N\int 1\,\mathrm{d}\theta - mg\int \cos\theta\,\mathrm{d}\theta\right),$$

$$\dot{\theta}^2 + A = \frac{2P\theta}{mR} - \frac{2\mu N\theta}{mR} - \frac{2mg\sin\theta}{mR} + B,$$

where B is an arbitrary constant,

$$\dot{\theta}^2 = \frac{2P\theta}{mR} - \frac{2\mu N\theta}{mR} - \frac{2g\sin\theta}{R} + B - A,$$

$$\dot{\theta}^2 = \frac{2P\theta}{mR} - \frac{2\mu N\theta}{mR} - \frac{2g\sin\theta}{R} + C, \tag{18}$$

where $C = B - A$.

Solving the initial condition problem to find C, we use the fact that at $t = 0$, angle $\theta = 0$, thus $\dot{\theta} = \ddot{\theta} = 0$. This renders $C = 0$ in equation (18), and so, we have

$$\dot{\theta}^2 = \frac{2P\theta}{mR} - \frac{2\mu N\theta}{mR} - \frac{2g\sin\theta}{R}. \tag{19}$$

Note, we now have obtained an expression for $\dot{\theta}^2$ which already appeared in equation (13). We can, therefore, substitute equation (19) in (13), and we obtain:

$$-mR\left(\frac{2P\theta}{mR} - \frac{2\mu N\theta}{mR} - \frac{2g\sin\theta}{R}\right) = N - mg\sin\theta. \tag{20}$$

Expanding and rearranging this, we get

$$N - mg\sin\theta = -2P\theta + 2\mu N\theta + 2mg\sin\theta,$$

$$N - 2\mu N\theta = -2P\theta + 2mg\sin\theta + mg\sin\theta,$$

$$N(1-2\mu\theta) = -2P\theta + 3mg\sin\theta,$$

$$N = \frac{3mg\sin\theta - 2P\theta}{1-2\mu\theta}. \tag{21}$$

So, now we have an expression of N in terms of the gravitational constant g, the variables m, μ, and P, and the more reasonable θ instead of $\dot{\theta}^2$.

And so, if we want to calculate when a mass would be slung out of its orbital path, we write $N = 0$ as this means, in physical terms, that the mass isn't resting on the cylinder anymore (since it doesn't exert a normal force on the mass). In other words, find the roots of equation (21) to find the one unknown variable. Note, R does not play a role. Of course, bear in mind that m is a point mass.